# Level-statistics of Disordered Systems: a single parametric formulation

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We find that the statistics of levels undergoing metal-insulator transition in systems with multi-parametric Gaussian disorders behaves in a way similar to that of the single parametric Brownian ensembles [1]. The latter appear during a Poisson → Wigner-Dyson transition, driven by a random perturbation. The analogy provides the analytical evidence for the single parameter scaling behaviour in disordered systems as well as a tool to obtain the level-correlations at the critical point for a wide range of disorders.

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The nature of the eigenfunctions can significantly affect the statistical behaviour of energy levels of a disordered system and thereby related physical properties e.g transport. The presence of disorder may cause localized waves in the system, implying lack of interaction between certain parts. This is reflected in the structure of the Hamiltonian matrix which is sparse or banded in the site representation depending on the dimensionality of the system. The variation of the disorder-strength can lead to a metal-insulator transition (MIT), with eigenfuctions changing from a fully extended state (metal) to a strongly localized one (insulator) with partial localization in the critical region. The associated Hamiltonian also undergoes a transition from a full matrix to a sparse or banded form and finally to a diagonal matrix. The statistical studies of levels for various types of disorders require, therefore, analysis of different ensembles. Here the nature of the localization and its strength is reflected in the measure and the sparcity of the ensemble, respectively. Our objective in this paper is to obtain a mathematical formulation for the level-correlations, common to a large class of disorders (Gaussian type); the information about the nature of disorder enters in the formulation through a parameter, basically a function of various parameters influencing the localization.

Recently it was shown that the eigenvalue distributions of various ensembles, with a Gaussian measure, appear as non-equilibrium stages of a Brownian type diffusion process [2]. Here the eigenvalues evolve with respect to a parameter related to the complexity of the system represented by the ensemble. The solution of the diffusion equation for a given value of the parameter gives, therefore, the distribution of the eigenvalues, and thereby their correlations, for the corresponding system. The present study uses the technique to analyze the spectral properties of the levels undergoing MIT in various disordered systems.

The Anderson model for a disordered system is described by a d-dimensional disordered lattice, of size $L$, with a Hamiltonian $H = \sum_n \epsilon_n a_n^+ a_n - \sum_{n\neq m} b_{mn}(a_n^+ a_m + a_n a_m^+)$ in tight-binding approximation. The site energies $\epsilon_n$, measured in units of the overlap integral between adjacent sites, correspond to the random potential. The hopping is assumed to connect only the $z$ nearest-neighbors (referred by $m$) of each site. In the site representation, $H$ turns out to be a sparse matrix of size $N = L^d$ with diagonal matrix elements $H_{kk} = \epsilon_k$ and off-diagonals $H_{kl}$ describing the hopping. The level-statistics can therefore be studied by analyzing the properties of an ensemble of (i) sparse real symmetric matrices, in presence of a time-reversal symmetry and (ii) sparse complex Hermitian matrices in absence of a time-reversal.

The MIT can be brought about by a competitive variation of the disorder and hopping rate which can be mimicked by a change of the distribution parameters of the ensemble measure $\rho(H)$. In this paper, we consider the case in which the site-energies $\epsilon_i$ are independent Gaussian distributions. The hopping can be chosen to be non-random or random (Gaussian). The $\rho(H)$, for any intermediate state of MIT can therefore be described by

$$\rho(H,y,b) = C\exp[-\sum_{s=1}^{\beta}\sum_{k\leq l}(1/2h_{kl;s})(H_{kl;s}-b_{kl;s})^2] \quad (1)$$

with subscript "$s$" of a variable referring to its components, $\beta$ as their total number ($\beta = 1$ for real variable, $\beta = 2$ for the complex one), $C$ as the normalization constant, $h$ as the set of the variances $h_{kl;s} = < H_{kl;s}^2 >$ and $b$ as the set of all $b_{kl;s}$. As obvious, in the limit $h_{kl;1}, h_{kl;2} \to 0$, eq.(1) corresponds to the non-random nature of $H_{kl}$ ($H_{kl} = b_{kl;1} + ib_{kl;2}$). Here the strongly insulated state corresponds to all $h_{kl} \to 0$ ($k \neq l$) and $b_{kl} \to 0$ (all $k,l$), implying no overlap between site states. The metal, with almost similar overlap between various site states, can be modeled by eq.(1) with all $h_{kl} \to \gamma^{-1}$, that is, a Wigner Dyson (WD) ensemble [1].

The eigenvalue distribution for a metal is given by the WD distribution, and, for an insulator by a Poisson distribution [3]. The distribution for various transition stages can be obtained by integrating $\rho$ over the associated eigenvector space. Let $P(\mu, h, b)$ be the probability of finding eigenvalues $\lambda_i$ of $H$ between $\mu_i$ and $\mu_i + \mathrm{d}\mu_i$ at a given $h$ and $b$, it can be expressed as $P(\mu, h, b) = \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i)\rho(H, h, b)\mathrm{d}H$. As discussed in ref. [2], it is possible to define a "complexity" parameter $Y$, a function of various distribution parameters $h_{kl;s}$ and $b_{kl;s}$ [2],

$$Y = -\frac{1}{2M\gamma}\ln\left[\prod_{k\le l}^{\prime}\prod_{s=1}^{\beta}|x_{kl;s}| \quad |b_{kl;s}|^2\right] + C \tag{2}$$

(here the $\prod'$ implies a product over non-zero $b_{kl;s}$ and $x_{kl;s} \equiv 1 - \gamma g_{kl} h_{kl;s}$ only, $g_{kl} = 2 - \delta_{kl}$, $C$ as a constant determined by initial ensemble and $M$ as the number of all non-zero parameters $x_{kl;s}$ and $b_{kl;s}$) such that the evolution of $P$ with respect to $Y$ results in the diffusion of eigenvalues with a finite drift due to their mutual repulsion,

$$\frac{\partial P}{\partial Y} = \sum_n \frac{\partial}{\partial \mu_n}\left[\frac{\partial}{\partial \mu_n} + \sum_{m\neq n}\frac{\beta}{\mu_m - \mu_n} + \gamma\mu_n\right] P \tag{3}$$

($\beta = 1, 2$ for Hamiltonians with and without time-reversal, respectively). Here $\gamma$ is an arbitray parameter [2]. The evolution reaches a steady state when $\partial P/\partial Y \to 0$; $P(\mu)$ in this limit is given by a WD distribution, $P(\mu) = \prod_{i<j}|\mu_i - \mu_j|^{\beta}\mathrm{e}^{-\frac{\gamma}{2}\sum_k \mu_k^2}$.

The spectral correlations for MIT can now be obtained by using following analogy. The eq.(3) is same as the equation governing an evolution of the distribution $P$ of the eigenvalues, of a Hamiltonian $H = \sqrt{f}(H_0 + \lambda V)$ with $\lambda^2 = \mathrm{e}^{2\gamma(Y-Y_0)} - 1$ and $f = \mathrm{e}^{-2\gamma(Y-Y_0)}$, from an initial state $H_0$ to WD ensemble. The transition, referred as WDT later on, is caused by a random perturbation $V$, taken from a WD ensemble (of variance $\gamma$), and of strength $\lambda$ [1,4]; the elements $H_{kl}$ are thus Gaussian distributed with a variance $h_{kl} = (1-f)/2\gamma$, $h_{kk} = 1/2\gamma$ and same mean for all of them. The transition (WDT) to equilibrium, with $Y - Y_0$ as the evolution parameter, is rapid, discontinuous for infinite dimensions of matrices [1]. But for small-$Y$ and large $N$, a smooth crossover can be seen in terms of a rescaled parameter $\Lambda = (Y - Y_0)/\Delta_\eta^2$, with $\Delta_\eta = \Delta N/\eta$ as the mean-level spacing in the correlated region of the spectrum at $Y - Y_0$, $\eta$ as the correlated "volume" and $\Delta \equiv \Delta(\mu, Y)$ as the mean level spacing of the whole spectrum. The intermediate states, corresponding to finite, non-zero $\Lambda$-values, during the crossover are known as Brownian ensembles (BE) [1,4] or Rosenzweig-Porter ensembles (RPE) [7,6,10]. The similar evolution equations of $P$ for MIT and WDT, imply a similarity in their eigenvalues distributions for all $Y$-values and thereby correlations for all $\Lambda$-values, under similar initial conditions (that is, $P(\mu, Y_0)$ same for both the cases although $\rho(H, Y_0)$ may be different). In finite disordered systems, therefore, a continuous family of intermediate statistics exists between metal and insulator which can be described by the BE for the corresponding $\Lambda$-value, occuring during Poisson $\to$ WD ensemble type WDT.

The behaviour of the mean level spacing $\Delta$ and the correlation volume $\eta$ divide the level-statistics for the large BE ($N \to \infty$) into three regions [7]:

(i) $N^2(Y - Y_0) \to 0$: $\Delta \propto N^{-1}$ and $\eta < N$, giving $\Lambda \to 0$ and Poisson statistics,

(ii) $N^2(Y - Y_0) \to \infty$: $\Delta \propto N^{-1/2}$ and $\eta > N$ giving $\Lambda \to \infty$ and WD statistics.

(iii)$N^2(Y - Y_0) = 1/2c$**, with** $c$ **as an arbitrary constant with respect to** $N$: although $\Delta$ still behaves as $o(1/N)$, but now $\eta \approx N$ thus giving $\Lambda = 1/2c\pi$ (referred as $\Lambda_{BE}^*$).

The $\Lambda = \Lambda_{BE}^*$ therefore corresponds to a third statistics, intermediate between Poisson and WD ensemble and is known as the critical Brownian ensemble (CBE). This being the case for arbitrary values of $c$, an infinite family of CBE occur during WDT. The presence of such a family can be seen from any of the fluctuation measures for WDT. One traditionaly used meausure in this regard is relative behaviour of the tail of nearest-neighbour spacing distribution $P(s)$, defined as $\alpha(\delta, \Lambda) = \int_0^\delta (P(s) - P_w(s))\mathrm{d}s / \int_0^\delta (P_p(s) - P_w(s))\mathrm{d}s$ with $\delta$ as any one of the crossing points of $P_w(s)$ and $P_p(s)$ (here subscript $w$ and $p$ refer to the WD case and Poisson case respectively). In the limit $N \to \infty$, $\alpha = 0$ and 1 for WDE and Poisson limit respectively. The FIG.1 shows the numerically obtained behaviour of $\alpha$ with respect to $|c - c^*|$ (for $\delta \approx 2.02$), with $c^*$ corresponding to one of the CBEs. The convergence of all the points on two branches for different $N$-values confirms the existence of a CBE at $c^*$ with a critical exponent $\nu \to \infty$ (as $\alpha(c) \approx \alpha(c^*) + constant.|c - c^*|N^{1/\nu}$ near $c^*$). The $N$-independence of $\alpha(c)$ also indicates the possibility of infinitely many $c^*$ (and its arbitrarity) and therefore an infinite family of CBEs.

For disordered systems with infinite system size $L$, the states are critical near a particular disorder $W^*$ (or energy), termed as critical point, with a correlation length $\zeta(W) \propto |W - W^*|^{-\nu}$, $\nu$ as the critical exponent. At the critical point, $\zeta \approx L$ and the critical value of the parameter $\Lambda$ (referred as $\Lambda_{AH}^*$) depends on the dimensionality; $\Lambda_{AH}^* = (Y - Y_0)/\Delta_\eta^2$, with both $Y$ and $\Delta_\eta$ dimensionality-dependent. A knowledge of $\Lambda_{AH}^*$ can then be used to map the critical level statistics at MIT for various dimensions $d > 2 \to \infty$ to the infinite family of critical Brownian ensembles (CBE).

The identification of the appropriate CBE corresponding to the critical Anderson Hamiltonian (CAH) re-

quires $\Lambda^*_{BE} = \Lambda^*_{AH}$, and, thus a prior knowledge of $\Lambda$-value associated with CAH. We consider here one example (this case is also used in our numerical analysis). Consider an Anderson system with the Gaussian disorder, same for each site, and random or non-random hopping between nearest neighbours. The corresponding ensemble measure can be described by eq.(1) with $h_{kk} = W^2/2$, $b_{kk} = 0$ and $h_{kl} = W_1^2/2$, $b_{kl} = t$ for $\{k,l\}$ pairs representing hopping, $h_{kl} \to 0$ and $b_{kl} \to 0$ for all $\{k,l\}$ values corresponding to dissconnected sites. This gives, by using eq.(2), $Y = -(N/2M\gamma)\ln\left[|1-\gamma W^2||1-2\gamma W_1^2|^{\beta z/2}|t+\delta_{t0}|^{\beta z}\right] + C$. Here $M = \beta N(N + z\delta_{t0} + 2 - \beta)/2$ with $zN$ as the number of the connected sites (nearest-neighbours) which depends on the topology and the dimensionality $d$ of the system. Analogous to WDT, a rescaling of $Y - Y_0$ would give the parameter for smooth transition: $\Lambda = 2(a - a_0)\beta^{-1}(\zeta/L)^d$ with $a(\alpha, t) \equiv \ln\left[|1-\gamma W^2||1-2\gamma W_1^2|^{\beta z/2}|t+\delta_{t0}|^{\beta z}\right]$, and $a_0$ as the value of $a$ at $Y_0$. Here $\Delta$ behaves as $O(N^{-1/2})$ (as $N^2(Y - Y_0) \to \infty$) and $\eta = \zeta^d$ with $\zeta$ as the localization length which depends on disorder $W$ and can be determined by using wavefunction statistics (e.g. inverse participation ratio). As disorder decreases, the $\zeta(W)$ increases resulting in a smooth increase in $\Lambda$ in finite systems, and, thereby the statistics intermediate between Poisson and WDE. However for infinite system sizes, $\Lambda \to 0$ and the statistics is Poisson as long as $\zeta < L$. But at a certain disorder strength, $\zeta \sim L$, which makes $\Lambda$ size-independent thus giving its critical value $\Lambda^*_{AH} \approx 2(a - a_0)\beta^{-1}$. The level statistics of the system at this disorder strength is given by the CBE with the same $\Lambda$-value and is critical due to (i) being different from both Poisson and WD behaviour, (ii) its invariance as $N \to \infty$. Now for a further decrease in disorder, $\zeta > L$, as a result $\Lambda \to \infty$, and the system has a WD statistics. However, for an infinitely long one dimensional lattice with $z$ nearest neghbours, the statistics always remains Poission irrespective of disorder strength. This is because $\zeta \approx z^2$ giving $\Lambda \approx z^2/L \to 0$ for $L \to \infty$ unless there is a very long range connectivity in the lattice (i.e $z \approx \sqrt{L}$). However a crossover from Poisson to WD ensemble can be seen for finite $L$ by varying the ratio $z^2/L$.

In general, $\Lambda$ will be a function of coordination number, disorder strength, hopping rate and dimensionality of the lattice as well as the level-density of the system. The boundary conditions/ topologies, leading to different sparcity and coordination numbers, may therefore result in different critical level statistics even if degree of disorder, hopping rate and dimensionality is same; this is in agreement with numerical observations [11] and analytical study for 2D systems [12]. Similarly different dimensions, can lead to diffenent local level-densities and thereby different critical points $\Lambda^*$.

Many results for the spectral fluctuations of the WDT with Poisson ensemble as an initial state are already known [5] and can directly be used for the corresponding measures for the MIT in different disordered systems. For example, consider the 2-level density correlator $R_2(r;\Lambda)$; $R_2(r;\Lambda) = <\nu(\mu_1,\Lambda)\nu(\mu_2,\Lambda)> / <\nu>^2 = \frac{N!}{(N-2)!}\int P(\mu,\Lambda)\mathrm{d}\mu_3..\mathrm{d}\mu_N$. Here $\nu(\mu,\Lambda) = N^{-1}\sum_i \delta(\mu - \mu_i)$ is the density of eigenvalues and $<..>$ implying the ensemble average. The $R_2$ for the Anderson transition in presence of a magnetic field, can therefore be given by the $R_2$ for WDT between Poisson $\to$ GUE ensemble [5,6]; $R_2 = 1 - Y_2$ where

$$Y_2(r;\Lambda) = -\frac{4\Lambda}{r}\int_0^\infty \mathrm{d}u\, F\, \mathrm{e}^{-2\Lambda u^2 - 4\pi\Lambda u} \qquad (4)$$

with $F = \sin(ur)f_1 - \cos(ur)f_2$, $f_1 = (2/z)[I_1(z) - \sqrt{8u/\pi}I_2(z)]$, $f_2 = (1/u)[I_2(z) - \sqrt{2u/\pi}I_3(z)]$, $z = \sqrt{32\pi\Lambda^2u^3}$ and $I_n$ as the $n^{\mathrm{th}}$ Bessel function. Here $R_2(r,\infty) = 1 - (\sin^2(\pi r)/\pi^2 r^2)$ and $R_2(r,0) = 1$ corresponding to metal and insulator regime respectively. A substitution of $\Lambda = \Lambda^*$ in eq.(4) will thus give the $R_2$ for the CAH. Similarly the nearest-neighbour spacing distribution $P(s)$ for the MIT can be given by using the one for the BE during Poisson $\to$ GUE transition [13]: $P(s;\Lambda) \propto (2\pi\Lambda)^{-1/2}s\mathrm{e}^{-s^2/8\Lambda}\int_0^\infty \mathrm{d}x \quad x^{-1}\mathrm{e}^{-x-x^2/8\Lambda}\sinh(xs/4\Lambda)$. Further the good agreement of the numerically obtained $P(s)$ for the CAH without time-reversal symmetry with that of a CBE (with same $\Lambda$-value) reconfirms our analytical result; the FIG.2(a,b) show the behaviour of the CAH in presence of a random hopping (CAH-I with $W = 8.15$, $W_1 = 1$, $t = 0$, and, thereby $\Lambda = 10.6$) and a non-Random hopping (CAH-II with $W = 21.3$, $W_1 = 0$, $t = 1$, thus $\Lambda \approx 0.53$), respectively along with their CBE analogs ($c$ determined by the relation $\Lambda^*_{AH} = \Lambda^*_{BE} = (2c\pi)^{-1}$); see [8] for the details on the numerics.

An important characterstic of critical level statistics is the level compressibility $\chi$, $\chi(\Lambda) \approx 1 - \int_{-\infty}^{\infty} Y_2(r;\Lambda)\mathrm{d}r$; $\chi = 0, 1$ in the metallic and the insulator phase, respectively, and takes an intermediate value at the hybrid phase near the critical point. The eq.(4) can be used to obtain $\chi(\Lambda) = 1 - 4\pi\Lambda\int_0^\infty \mathrm{d}u f_1(z)\exp[-2\Lambda u^2 - 4\pi\Lambda u]$. The critical region, with its finite $\Lambda$ value ($= \Lambda^*$), will thus have a level compressibility different from both metal and insulator regimes; a $0 < \chi < 1$ value is supposed to be an indicator of the multifractal nature of the eigenvectors.

The compressibility of the spectrum can also be seen from the "number variance" $\Sigma_2$ which describes the variance in the number of levels in an interval of $r$ mean level spacings. The $\Sigma_2(r)$, basically a measure of the "spectral rigidity" is related to the compressibility: $\lim r \to \infty$ $\Sigma_2 \approx \chi r$. The FIG.3 shows the numerical behaviour for the $\Sigma^2(r;\Lambda)$ for the CAH-I and CAH-II along with their CBE analogs; the good agreement in each case reverifies our claim about the similarity between the MIT

and WDT. Furthermore FIG.3. also shows the fractional behaviour of $\chi$ for CBE, which is in agreement with the $\chi$-form given in the preceeding paragraph.

The statistical measures for the Anderson transition in presence of a time-reversal symmetry can similarly be obtained by using their equivalence to WDT from Poisson $\rightarrow$ GOE ensemble. However due to the technical difficulties [5]), only some approximate results are known for the latter case. for example, the $R_2$ for small-$r$ can be given as $R_2(r,\Lambda) \approx (2r-1)^{1/2} J_{1/3}\left((2r-1)^{3/2}/3\Lambda\right) e^{r/2\Lambda}$ with $J(z)$ as the Bessel function (obtained by solving eq.(17) of [5]). Similarly for large-$r$ behaviour, $R_2$ can be shown to satisfy the relation (see eq.(21) of [5]) $R_2(r,\Lambda) = R_2(r,\infty) + 2\beta\Lambda \int_{-\infty}^{\infty} \mathrm{d}s R_2(r-s;0) - R_2(r-s;\infty)/(s^2+4\pi^2\beta^2\Lambda^2)$. By taking $\beta = 1$ and $R_2(r,\infty) = 1 - \sin^2(\pi r)/\pi^2 r^2 - \left(\int_r^{\infty} \mathrm{d}x \sin\pi x/\pi x\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\sin\pi r/\pi r\right)$ (GOE limit), the $R_2(r,\Lambda)$ for the transition in presence of a TRS can be obtained; $R_2(r,\Lambda) \approx R_2(r,\infty) + 4\Lambda/(r^2+4\pi^2\beta^2\Lambda^2)$ . The lack of the knowledge of $R_2(r,\Lambda)$ for entire energy-range handicaps us in providing an exact form of $\chi$. However the $P(s)$ for this case can be given by using the one for a BE (RPE) during Poisson $\rightarrow$GOE transition [13]: $P(s,\Lambda) = (\pi/8\Lambda)^{1/2} s e^{-s^2/16\Lambda} I_0(s^2/16\Lambda)$, with $I_0$ as the Bessel function.

In the end, a comparison of our results with some of the past studies is crucial. The presence of a fractional compressibility in an ensemble essentially same as CBE and its possibility as a model for CAH was also suggested in [9] which was later on contradicted in [10]. The claim in [10] is disproved by our numerical studies of both the transitions, showing similar behaviour for compressibility, besides other fluctuation measures at the critical point. The numerical work therefore reaffirms our claim based on the exact analytical work: (i) the level-statistics for MIT in disordered systems with a Gaussian disorder can be described by the same for WDT with a Poisson initial condition, (ii) the level-statistics in the disordered systems is indeed governed by a single scaling parameter.

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## I. FIGURE CAPTION

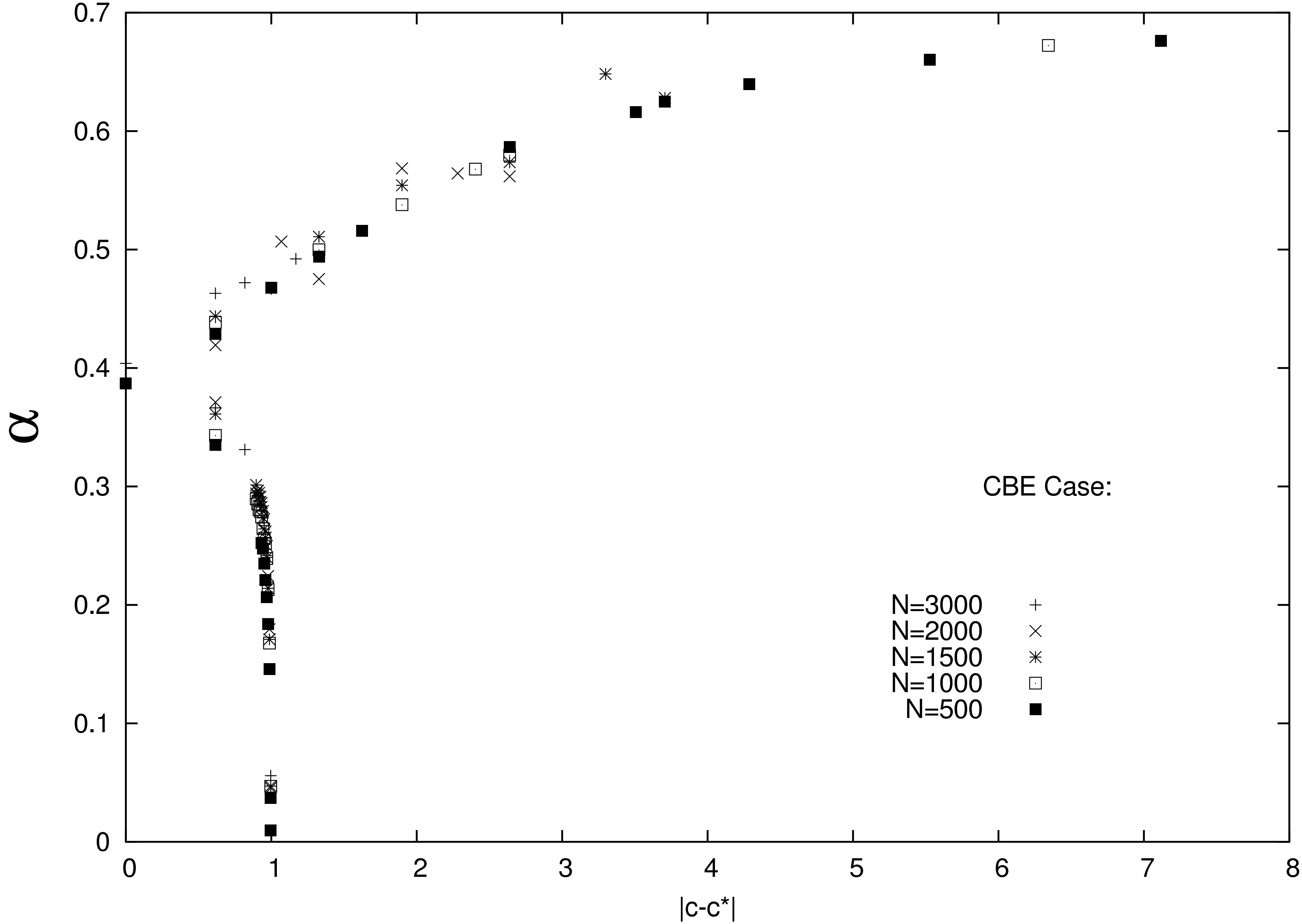
### CAPTIONS

Fig. 1. $\alpha$ vs $|c - c^*|$ for WDT

Fig. 2. $P(S)$ vs $S$ for (A) CAH-I and the corresponding CBE ($c = 0.015$), (b) CAH-II and the CBE with $c = 0.3$.

Fig. 3. $\Sigma^2$ vs $r$ for (i) CAH-I and CBE with $c = 0.3$, (ii) CAH-II and the CBE with $c = 0.015$.

Figure 1



α
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
8
|c-c*|
CBE Case:
N=3000
N=2000
N=1500
N=1000
N=500

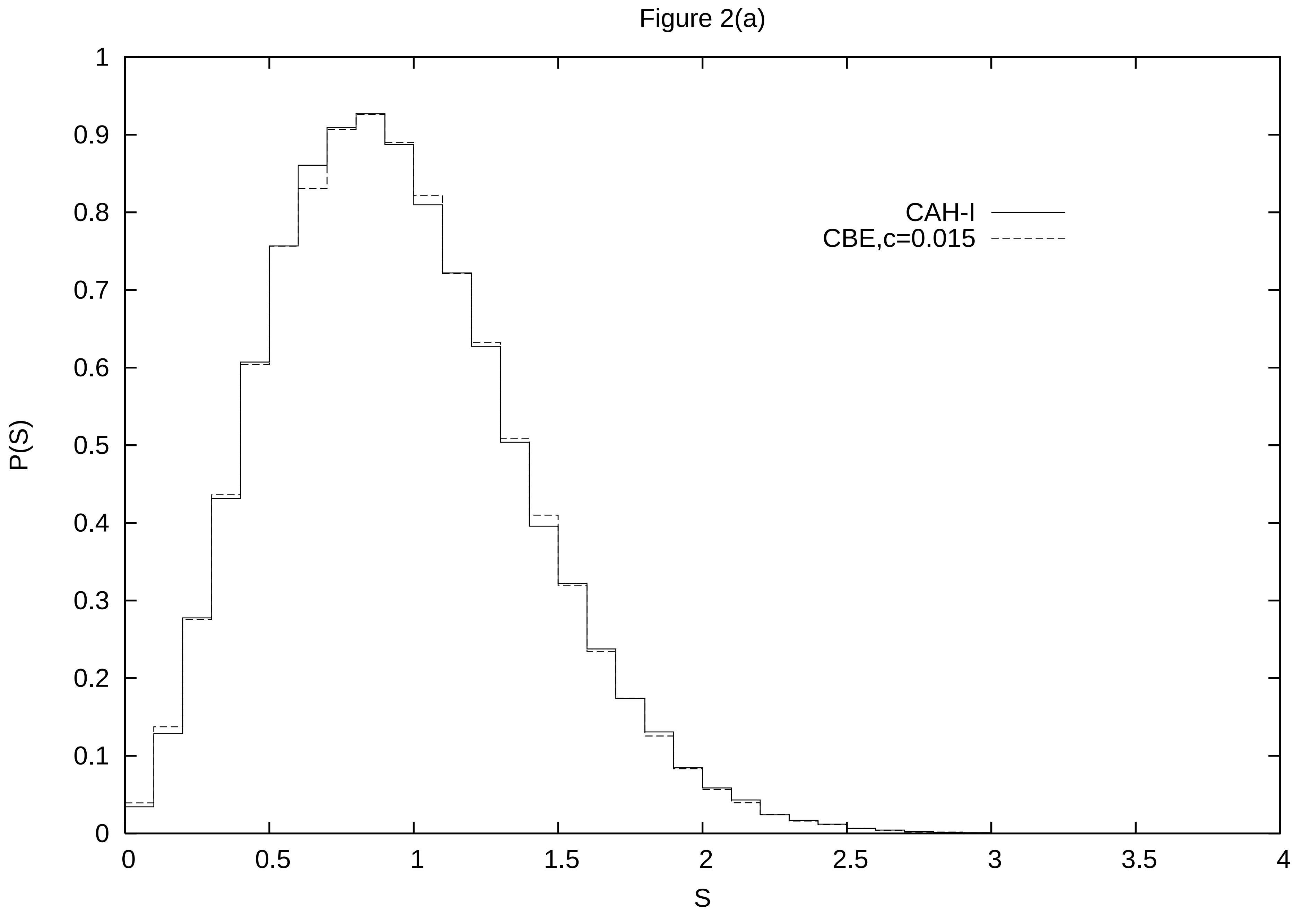


Figure 2(a)
CAH-I
CBE,c=0.015
P(S)
S
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3
3.5
4

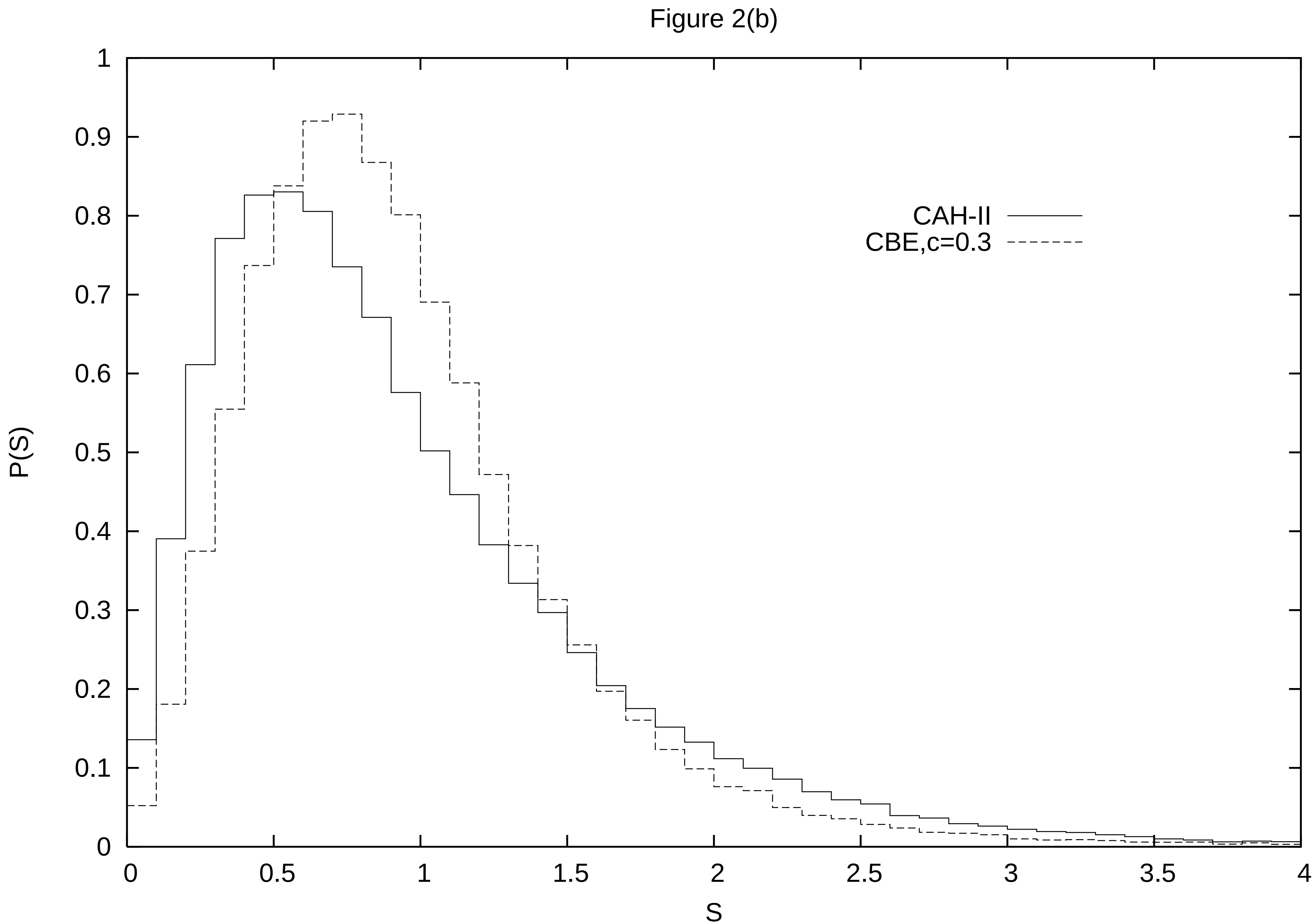


Figure 2(b)
CAH-II
CBE,c=0.3
P(S)
S
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3
3.5
4

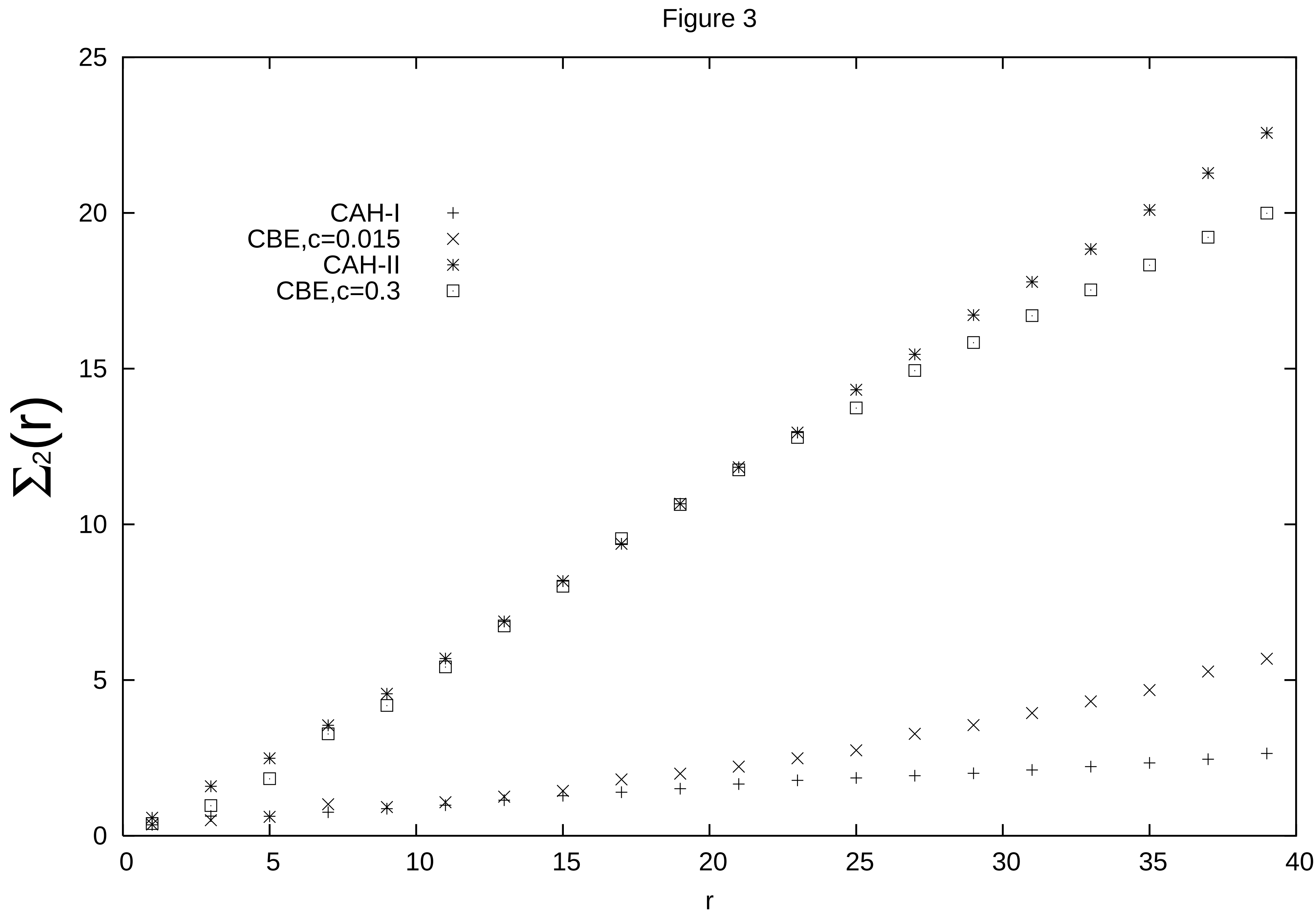


Figure 3
$\Sigma_2(r)$
r
CAH-I
CBE,c=0.015
CAH-II
CBE,c=0.3
0
5
10
15
20
25
30
35
40